Hub Sync in Heterogeneous Networks
Synchronization goes upscale

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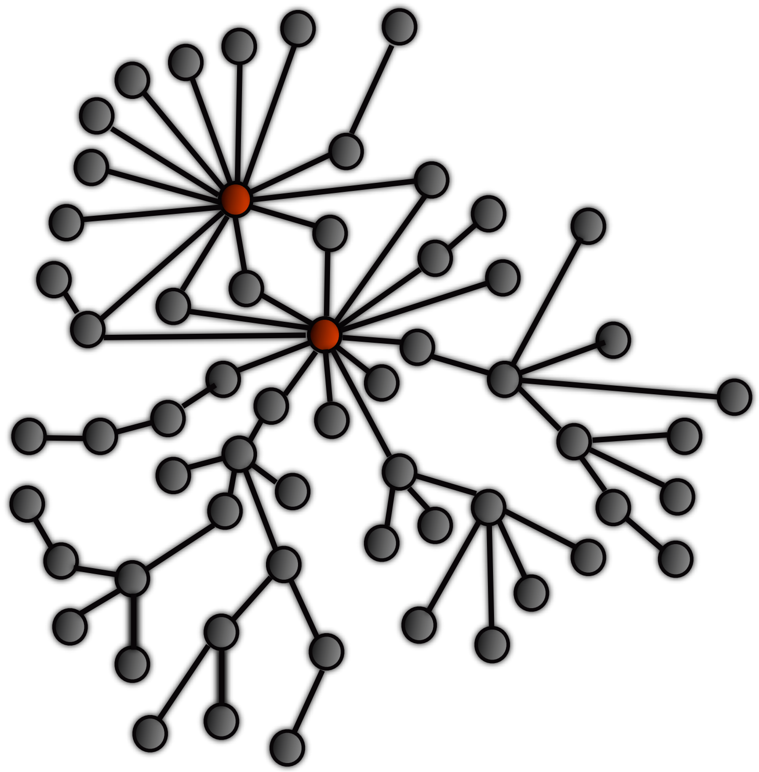
How the individual dynamics and the interaction structure affect the network function



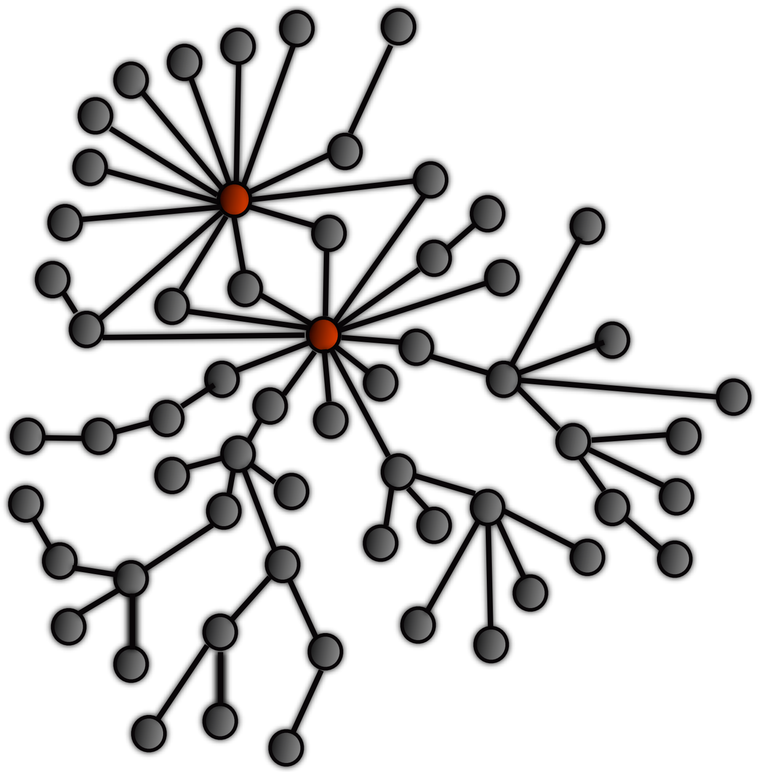
Typical structure of real networks

Albert and Barabási, Rev. Mod. Phys. 74, 47 (2002).

Chung and Lu, AMS (2006).

Barrat, Barthelemi, Vespegnani, Cambridge UP (2008).

Hubs are ubiquitous



Scientific Collaboration

Airports/Cities

Internet/www

Cortex

Background: Global Synchronization

Heterogeneity hinders global synchronization

McGraw & Menzinger, Phys. Rev. E 72, 015101 (2005).

Hwang, et al. , Phys. Rev. Lett. 94, 138701 (2005)

Nishikawa, et al., Phys. Rev. Lett. 91, 014101 (2003)

Hubs play a central role

hub neurons play a major role during development
- orchestrate the collective behavior

Bonifazi, Goldin, Picardo, et al., Science 326, 1419 (2009).

Hubs play a central role

hub neurons can drive towards epileptic seizures

Morgan and Soltesz, PNAS 105, 6179 (2008).

Hubs play a central role

Numerical experiments suggest

Heterogeneity favors the formation of Synchronization Cluster

C. Zhou and J. Kurths, Chaos 16, 015104 (2006);

Gardenes, Moreno, Arenas, Phys. Rev. Lett. 98, 034101 (2007);

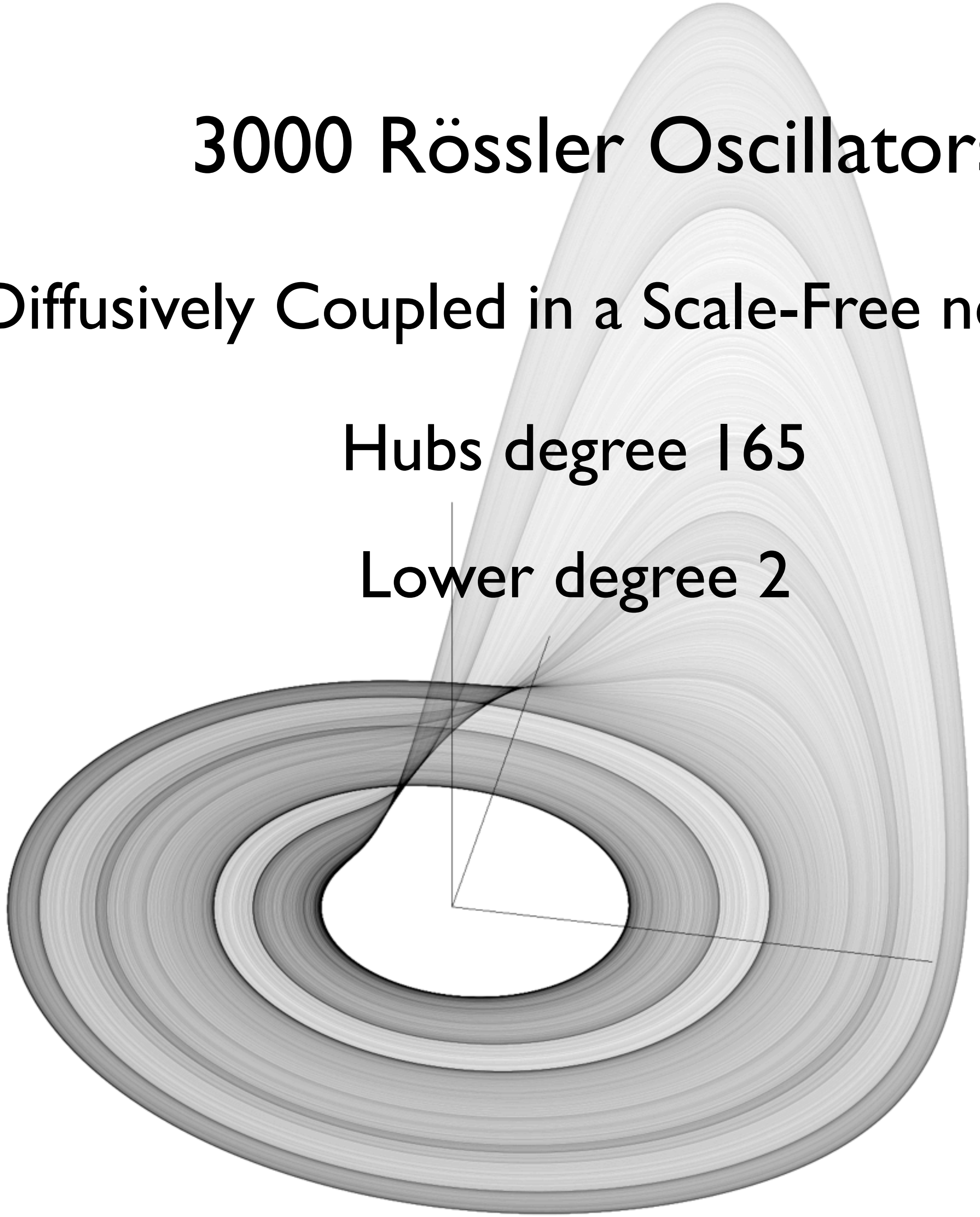
Pereira, Phys. Rev. E 82 036201 (2010)

3000 Rössler Oscillators

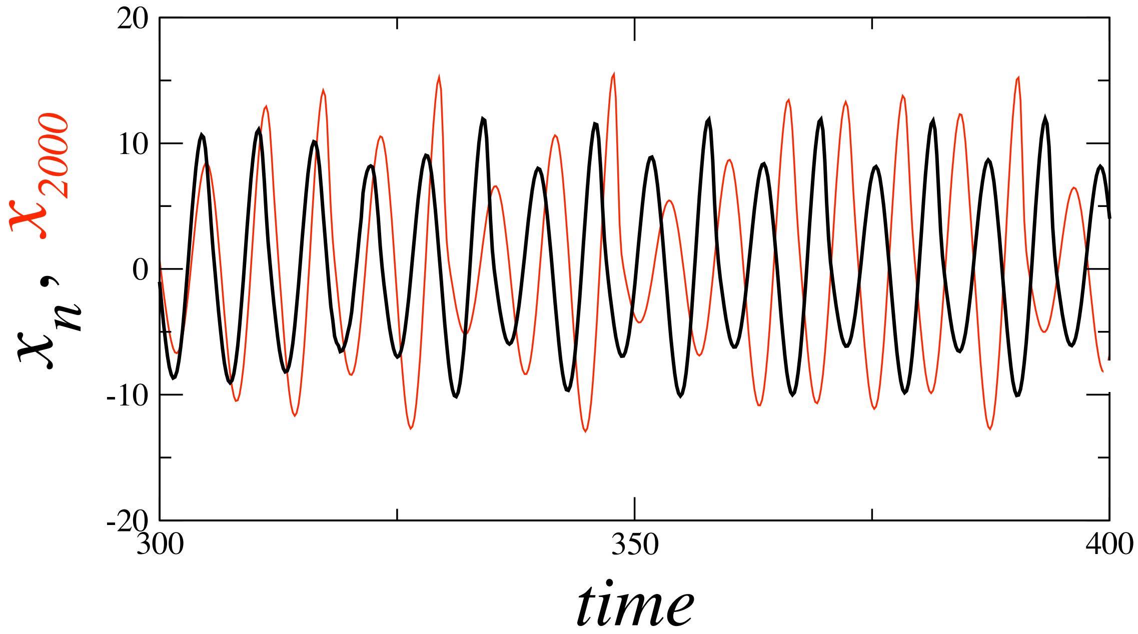
Diffusively Coupled in a Scale-Free network

Hubs degree 165

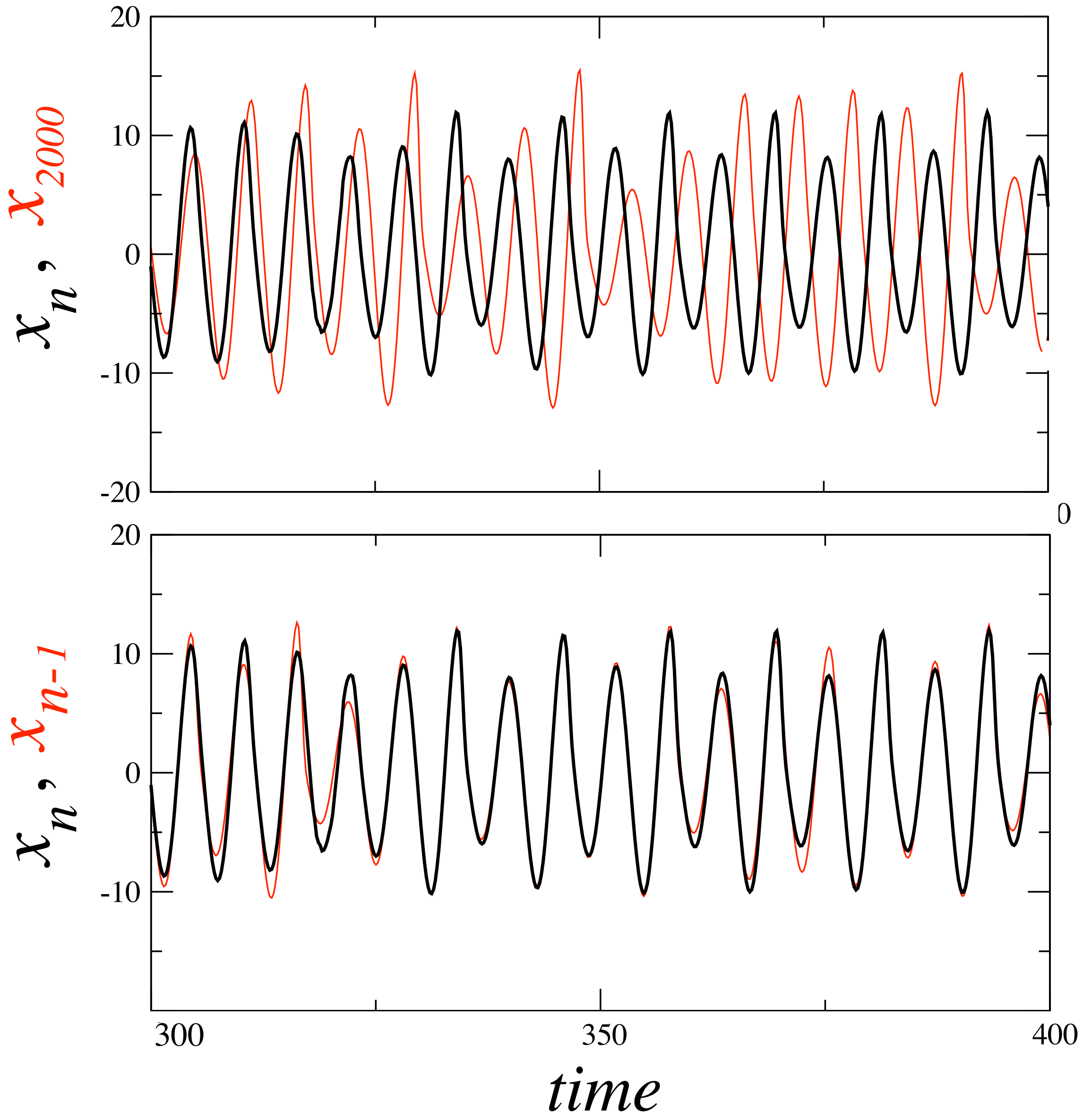
Lower degree 2



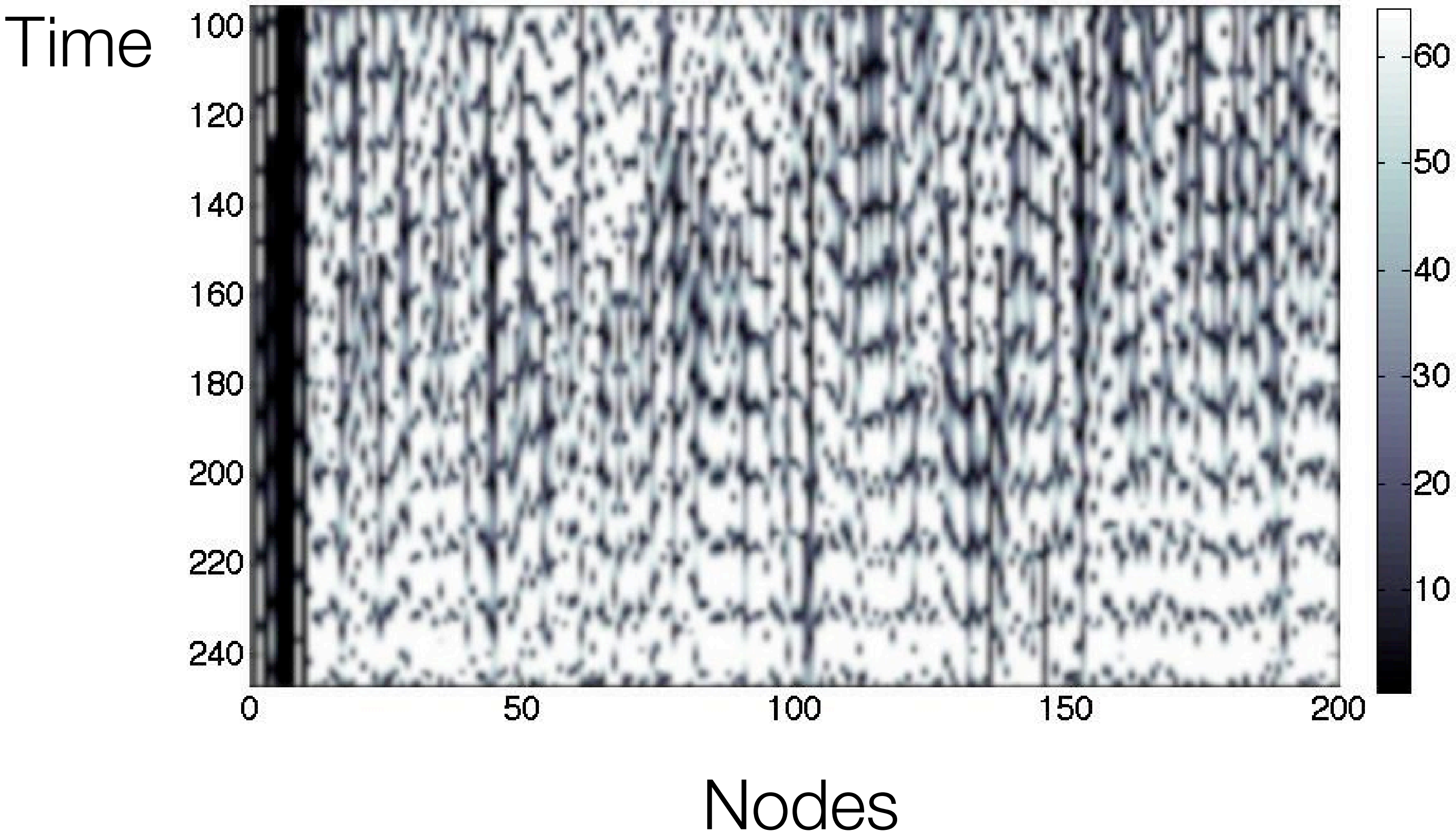
Time Series of the Roesslers



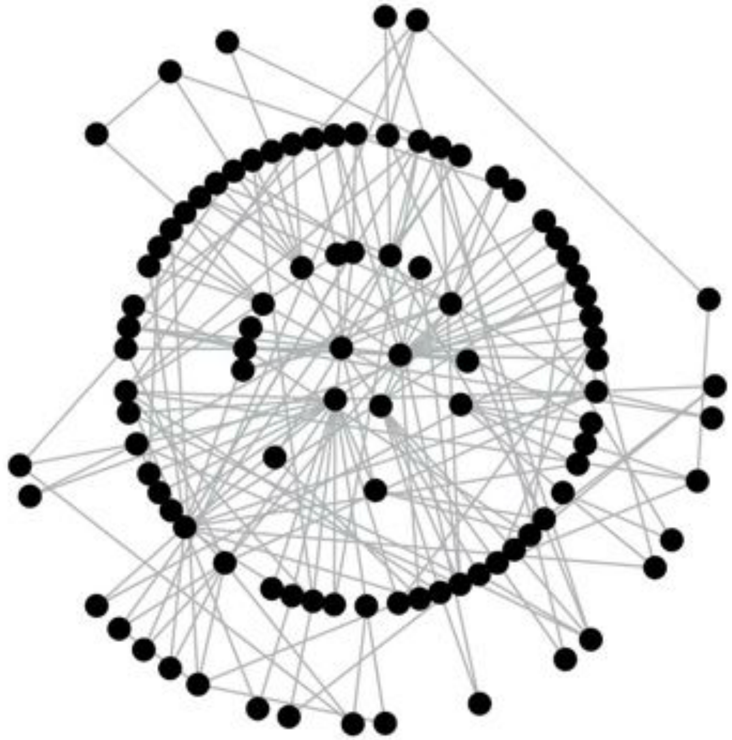
Time Series of the Roesslers



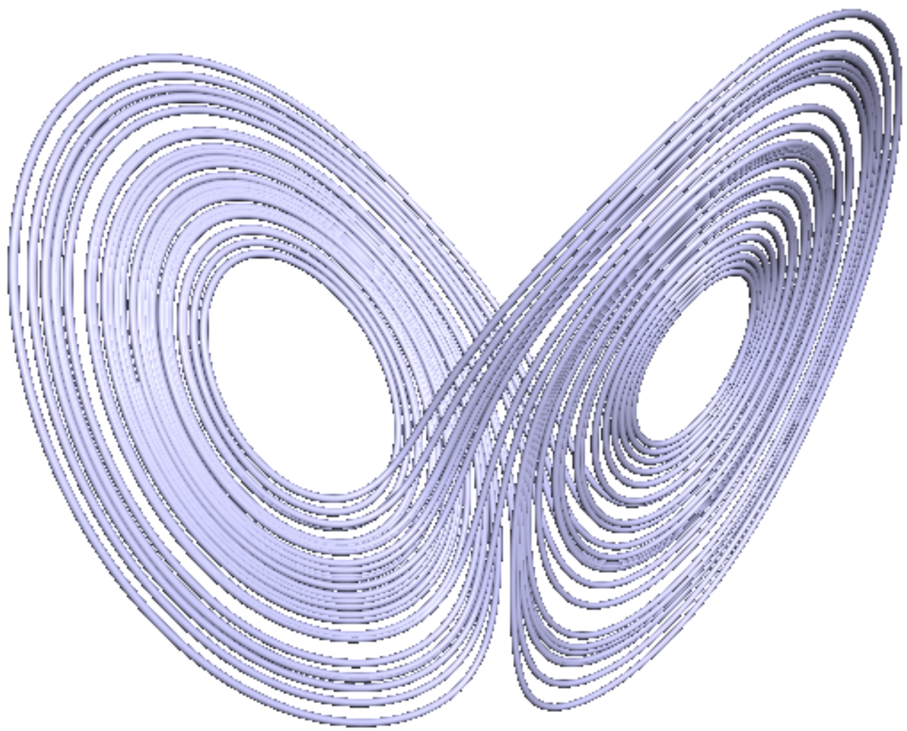
Synchronized Nodes



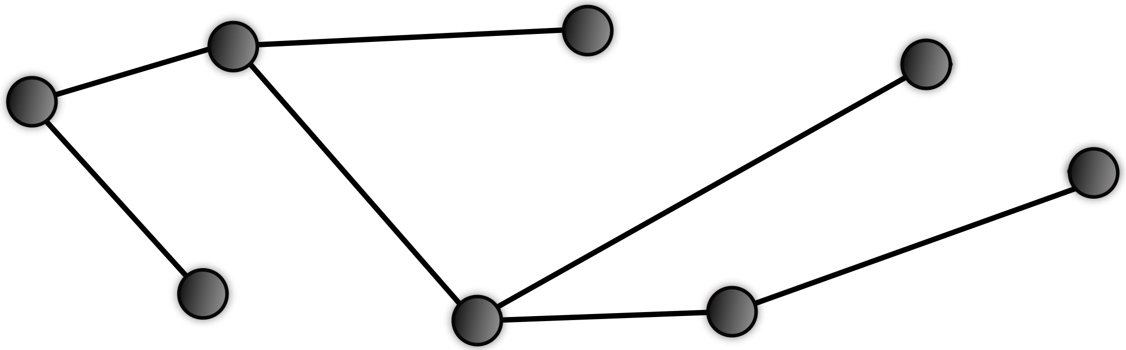
Synchronization Structure



Lorenz Dynamics



Lorenz Dynamics: Heterogeneous Network

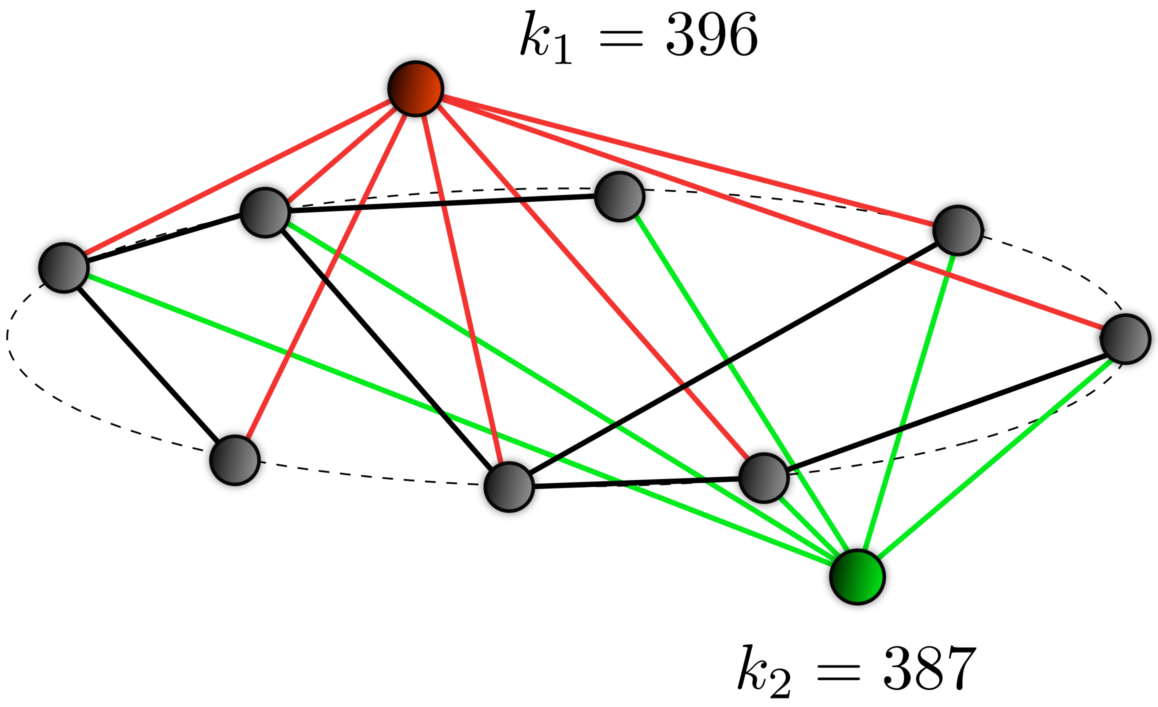


$p = 0.02$

Start with a random network

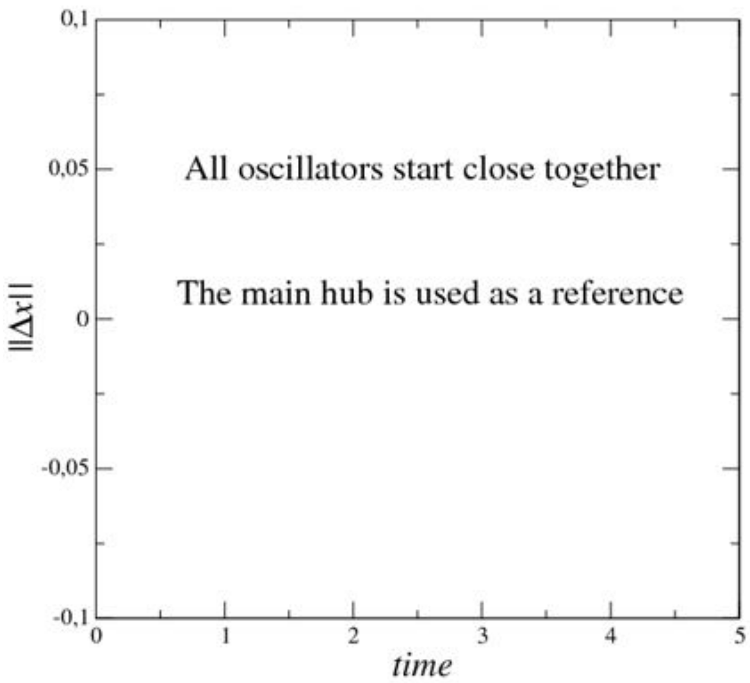
400 Nodes

Lorenz Dynamics: Heterogeneous Network

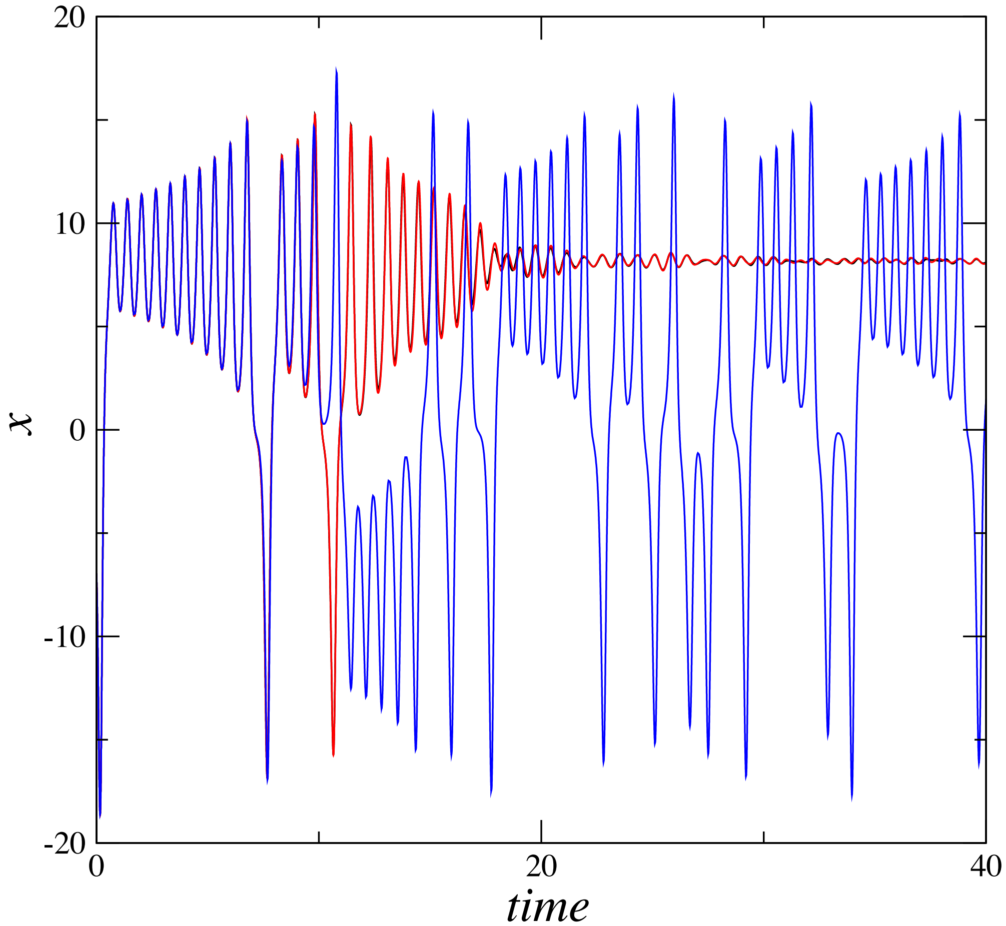


Include the Hubs randomly

Difference between of nodes and the main Hub



Time Series of the Lorenz: Hubs and Low-degree



Network Model

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}(\mathbf{x}_i) + \frac{\alpha}{m}\sum_{j=1}^{n} A_{ij}(\mathbf{x}_j - \mathbf{x}_i)$$

m	Largest Degree
$\mathbf{A}$	Adjacency Matrix

Network Model

Network Assumptions: Heterogeneity

$$\frac{k_i}{m} = \mu_i \left[1 + O\left(\frac{1}{m^\gamma}\right)\right]$$

Hubs

$$\frac{k_i}{m} = O\left(\frac{1}{m^\gamma}\right)$$

Low degree Nodes

Network Model

Dynamics Assumptions: Stability

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

and consider the flow ϕ^t

Network Model

Dynamics Assumptions: Stability

There is an attractor $U \subset \mathbb{R}^m$

$$\phi^t(U) \subset U$$

Network Model

Dynamics Assumptions: Stability

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y}) + \mathbf{g}(\mathbf{y}, t)$$

such that $\|\mathbf{g}(\mathbf{y}, t)\| \leq \varepsilon$

Consider the flow ψ^t

Network Model

Consider the Borel measure ν supported on U

$$\left\| \int h d(\phi^t_* \nu) - \int h d(\psi^t_* \nu) \right\| \leq M\varepsilon$$

for all smooth functions h

Main Results

Consider the diffusively coupled model under satisfying the two assumptions. Let all initial conditions be independent and identically distributed according to a measure ν *supported on* U*. Let* i *and* j *stand for hubs and let* $\mu_i > \mu_j$ *and* $\alpha\mu_i > \alpha_c$ *with*

$$\mu_i - \mu_j > O(m^{-\gamma})$$

then

$$\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| \leq K \frac{\alpha(\mu_i - \mu_j)}{\mu_i \alpha - \alpha_c},$$

with probability $1 - O(m^{-\gamma})$ *and* $K = K(\boldsymbol{f})$*.*

Main Ideas: Mean Field Reduction

Coupling term in the Hubs

$$\frac{1}{m}\sum_{j=1}^{n} A_{ij}(\mathbf{x}_j - \mathbf{x}_i) = \underbrace{\mathbf{v}_i(t)}_{local\ \ mean\ \ field} - \frac{k_i}{m}\mathbf{x}_i$$

Main Ideas: Mean Field Reduction

Initially the measure of the network is product

$$\nu^n$$

Main Ideas: Mean Field Reduction

1. Step

Obtain the evolution of ν^n

Main Ideas: Mean Field Reduction

Consider the network flow

$$\Psi^t : \mathbb{R} \times \mathbb{R}^{nm} \to \mathbb{R}^{nm}$$

Main Ideas: Mean Field Reduction

Hence

$$\Psi^t_*(\nu^n)$$

is the measure for the network

Main Ideas: Mean Field Reduction

Loosely Speaking

$$\Psi_*^t(\nu^n) \approx \underbrace{\lambda^t}_{hubs} \times \underbrace{[\omega^t]^{n-\ell}}_{low\,deg}$$

Low deg are almost independent

Main Ideas: Mean Field Reduction

Loosely Speaking

$$\Psi_*^t(\nu^n) \approx \underbrace{\lambda^t}_{hubs} \times \underbrace{[\omega^t]^{n-\ell}}_{low\,deg}$$

Take expectations with all the hubs coordinates and time fixed

Main Ideas: Mean Field Reduction

Mean Field Theory: Uniformity

$$\mathbb{P}(\|\mathbf{v}_i(t) - \mu_i \mathbf{u}(t)\| \geq \varepsilon) = O\left(\frac{1}{m^\gamma}\right)$$

Main Ideas: Stability

Study the quantity

$$\xi = \mathbf{x}_j - \mathbf{x}_i$$

where i and j stand for hubs

Main Ideas: Stability

$$\frac{d\xi}{dt} = \mathbf{K}(t, \mu_i)\xi + \alpha\eta_i$$

where

$$\mathbf{K}(t, \mu_i) = \int_0^1 D\mathbf{F}(\mathbf{x}_i + \beta\xi)d\beta - \alpha\mu_i\mathbf{I}$$

Main Ideas: Stability

$$\frac{d\xi}{dt} = \mathbf{K}(t, \mu_i)\xi + \alpha\eta_i$$

where

$$E\eta_i = O(\mu_i - \mu_j) + O\left(\frac{1}{m^\gamma}\right)$$

Hub Sync Stability

1. Control the coupling so that the homogeneous equation is uniformly asymptotic stable

2. Variations of Parameters

Hub Sync Stability

Numerical study reveals that Hub synchronization it is still present on random networks without the stability assumption

Take Home

The rich socialize

Functional Matthew Effect

Take Home

Cheaper to Synchronize the Hubs

$\alpha_c = O(1)$ Hubs

$\alpha_c = O(m)$ Global

Take Home

Heterogeneity allows for the expression of a rich variety of collective phenomena