Quantifying edge-to-edge relations by failure induced flow redistribution

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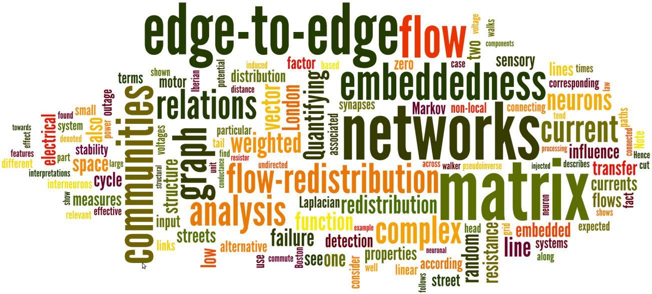
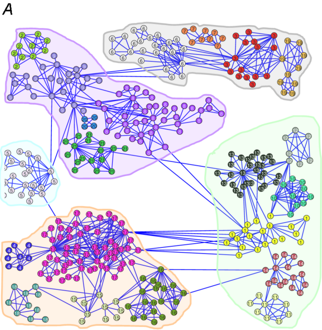
Outline

- Background: why edge relations?
- Edge relations through flow redistribution
- Using edge relations for network analysis

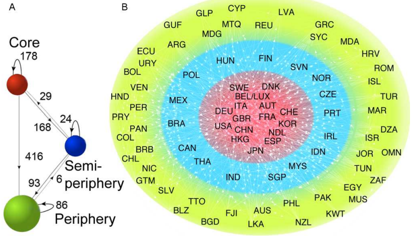


Background: why edge relations?

- Network Analysis so far mainly **node** centric
- Communities, node roles, centralities, etc.



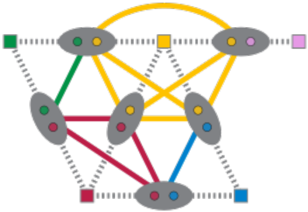
[Delvenne et al., 2010]



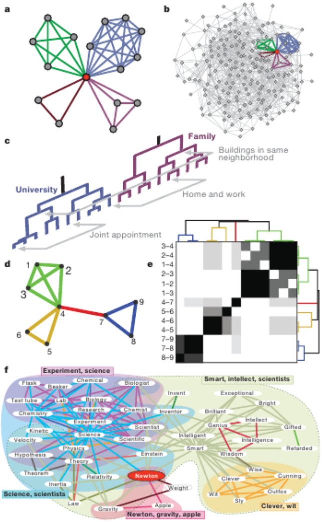
[Cooper et al 2010]

Some exceptions

- Line graph analysis (Evans & Lambiotte)
- Link Clustering (Ahn et al.)
- *Structural Edge Relationships*

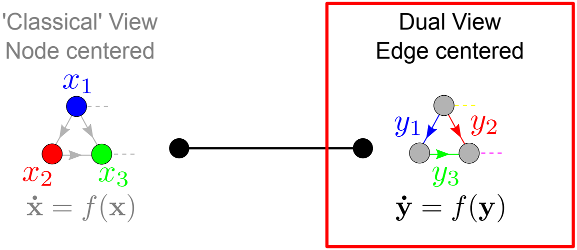


[Wikipedia]



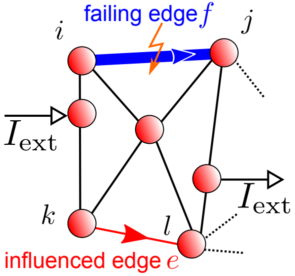
[Ahn et al. 2010]

Dual Perspective – Edge centered



- Circuit Theory: voltage *vs* currents
- Computational mechanics: displacement *vs* stress
- Optimization: Primal *vs* Dual variables
- Systems engineering, estimation theory, etc...

How to quantify edge relations? Flow redistribution!



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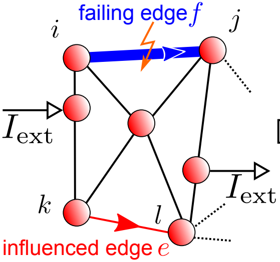
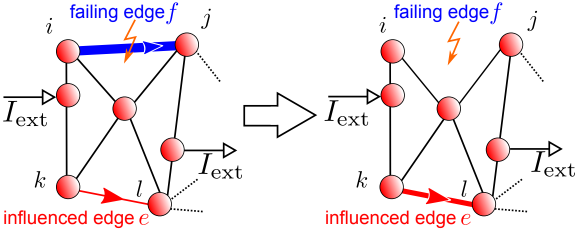
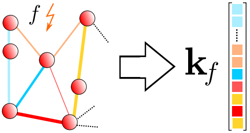


How to quantify edge relations? Flow redistribution!



Assuming a linear flow on the edges $L\mathbf{V} = \mathbf{I}_{\mathrm{ext}}$



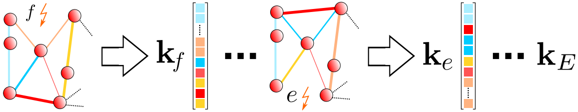
G=diag(g) incidence matrix incidence vector

$$\Delta_f \mathbf{i} = \left[\frac{G B^T L^\dagger \mathbf{b}_f}{1 - g_f\, \mathbf{b}_f^T L^\dagger \mathbf{b}_f} \right] \mathrm{i}_f \equiv \mathbf{k}_f\ \mathrm{i}_f$$

edge weight pseudoinverse of Laplacian

independent of current injections

The flow redistribution matrix



- Flow redistribution matrix

$$K_{E \times E} \equiv [\mathbf{k}_1 \cdots \mathbf{k}_E]$$

- Independent of current injections
- Describes **topological** feature of system in the edge space: **edge-to-edge coupling**

Characterising the flow redistribution matrix

The flow redistribution matrix can be decomposed

$$K = M\left[\mathrm{diag}(\boldsymbol{\varepsilon})\right]^{-1}$$

into the edge-to-edge transfer function matrix

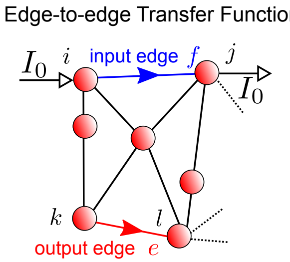
$$M_{E\times E} \equiv GB^{T}L^{\dagger}B$$

and the edge-embeddedness

$$\varepsilon_e \equiv 1 - g_e\,\mathbf{b}_e^{T}L^{\dagger}\mathbf{b}_e = 1 - g_e R_e$$

The edge to edge transfer function

$$M_{E\times E} \equiv GB^T L^\dagger B$$



- **Transfer function** – describes how input on edge translates into flow on other edges
- Physics interpretation – **discrete Green's function** (edge space)
- **Projection matrix** – into the weighted cut space of the graph

The edge to edge transfer function – spectral properties

$$M_{E\times E} \equiv G B^T L^\dagger B$$

- (idempotent) $M^2 = M$
- $N-1$ eigenvalues $\lambda = 1$
- $E-(N-1)$ eigenvalues $\lambda = 0$ — corresponds to number of independent cycles in network

Can be expressed in terms of resistance distances / commute times

$$\begin{aligned} M_{ef} &= \frac{g_e}{2}(R_{jk} - R_{ik} + R_{il} - R_{jl}) && (1) \\ &= \frac{\pi_e}{4}\left((T_{jk} - T_{ik}) + (T_{il} - T_{jl})\right) && (2) \end{aligned}$$

The edge-embeddedness

$$\varepsilon_e \equiv 1 - g_e\,\mathbf{b}_e^T L^\dagger \mathbf{b}_e = 1 - g_e R_e$$

R_e – resistance distance between endpoints of edge e

- related to the projection into the **cycle space**
- High embeddedness – edge features in many cycles (weighted)
- Zero embeddedness – edge defines a cut (disconnects the network)

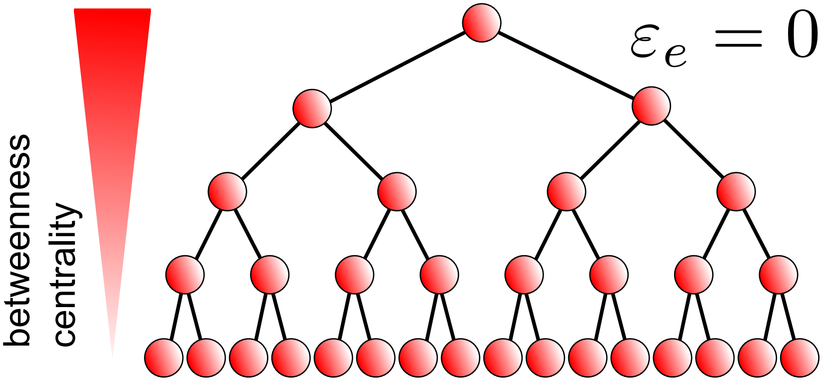
The edge-embeddedness II

$$\varepsilon_e = 1 - \pi_e \frac{T_{ij}}{2} = 1 - \frac{T_{ij}}{2\tau_e},$$

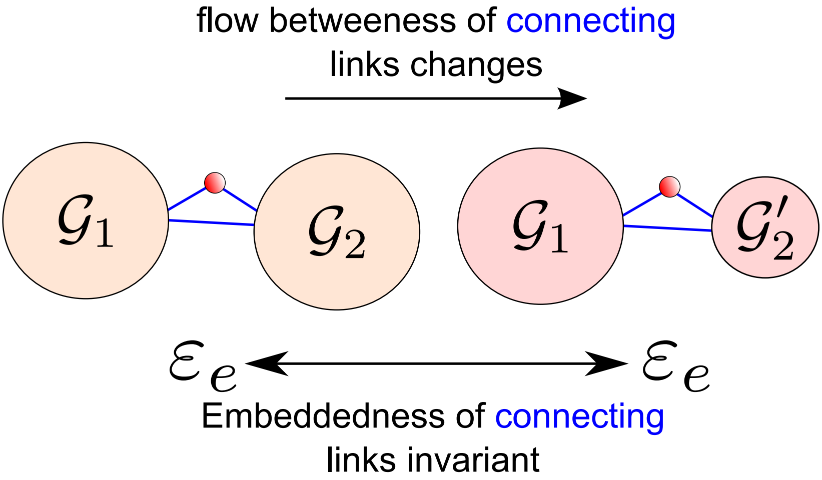
T_{ij} – commute time between nodes i, j (endpoints of edge e)

- Unweighted networks – probability of *not* finding the edge in a randomly selected spanning tree
- $\sum \varepsilon_e = \#$cycles in network
- Related to **graph sparsification**

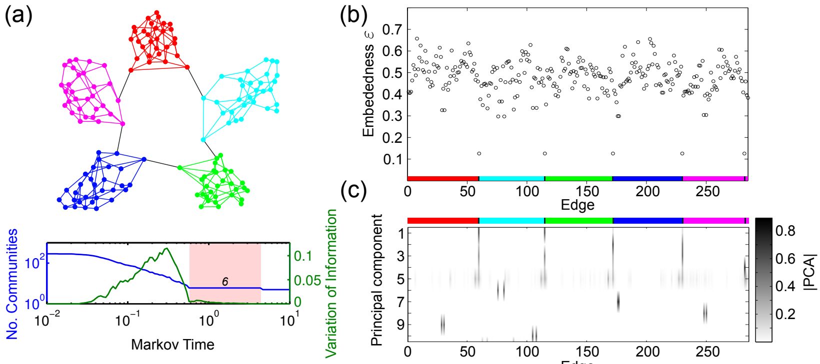
Embeddedness vs betweenness centrality



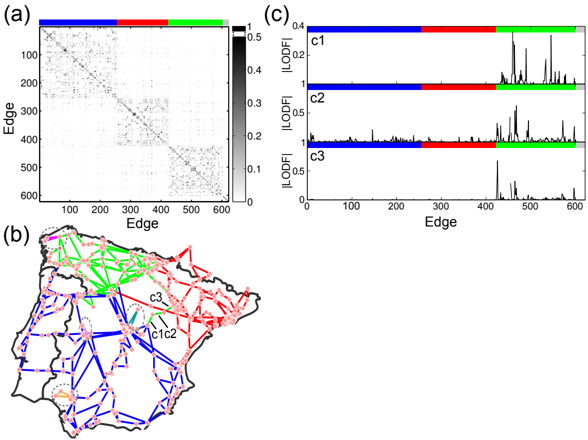
Embeddedness vs flow betweenness centrality



Applications: a toy example



Applications 1 – Iberian Power Grid



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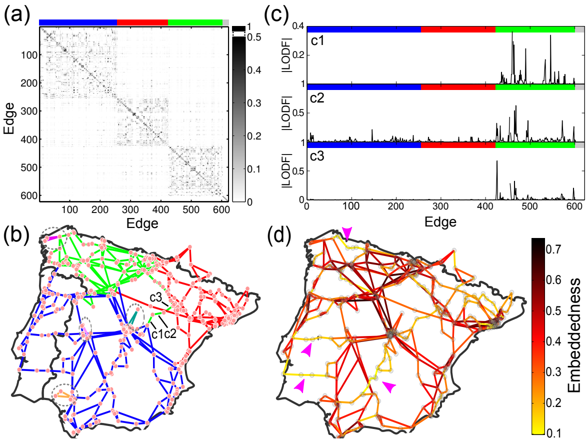

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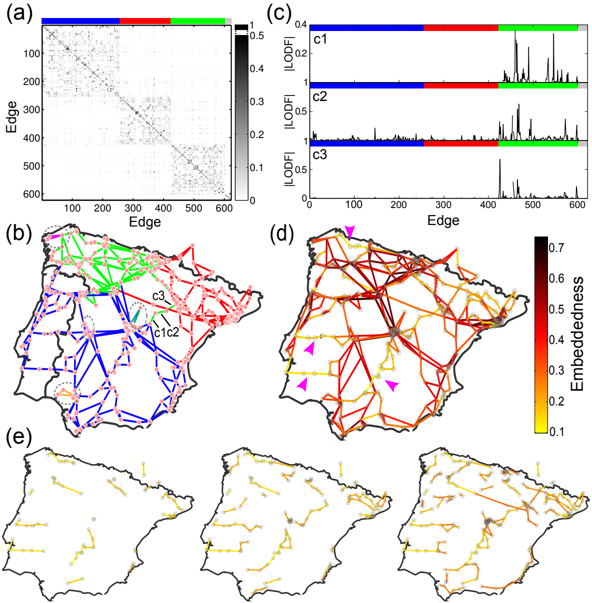
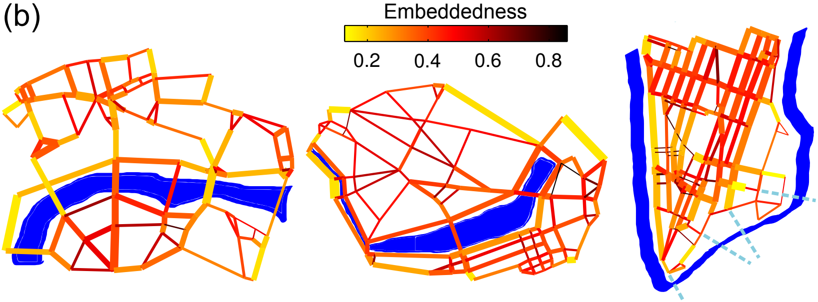


Applications 2 – Street networks



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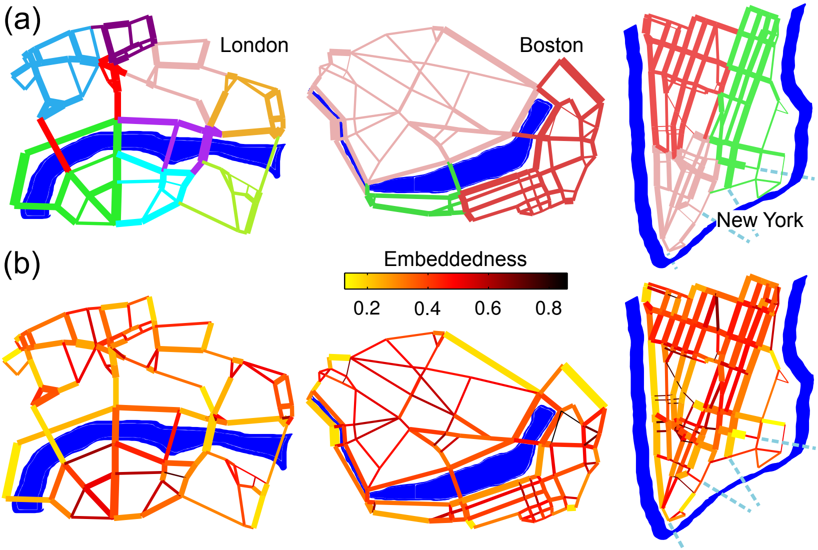
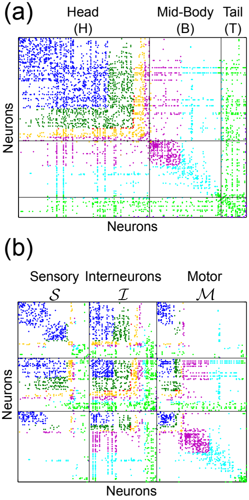
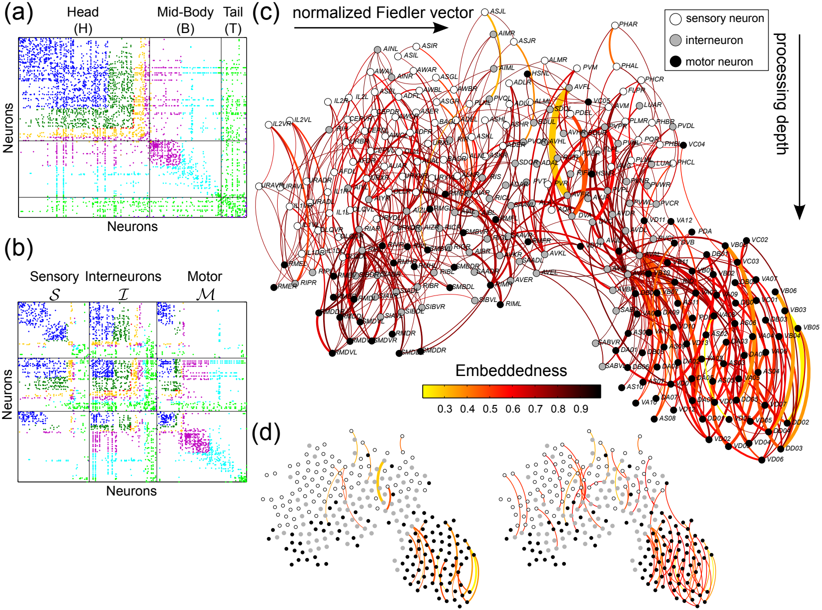


Applications 3 – C. elegans



Applications 3 – C. elegans



Take home messages

- Flow redistribution can characterise edge-to-edge relations
- Flow redistribution matrix – describes topological property in edge space
- Decomposable in measures with graph theoretic meaning:
  - Edge transfer function matrix (discrete Greens function)
  - Edge-embeddedness (projection into cycle space, sparsification)
- Ability to detect **non-local** effects in the edge coupling

Thank you!

The people..

- J. Lehmann (ABB)
- S. N. Yaliraki
- M. Barahona

The money...

- ONR
- EPSRC
- Studienstiftung des dt. Volkes

Everybody else

–

Thanks for listening!

Things left in the dark...

Schaub, M.T.; Lehmann, J.; Yaliraki, S.N. & Barahona, M., *Structure of complex networks: Quantifying edge-to-edge relations by failure-induced flow redistribution*, Network Science, April 2014, Vol. 2(1), pp. 66-89

QUESTIONS?