# Optimal Networks, Congestion and Braess' Paradox

Raúl J. Mondragón
Department of Electronic Engineering
Queen Mary University of London

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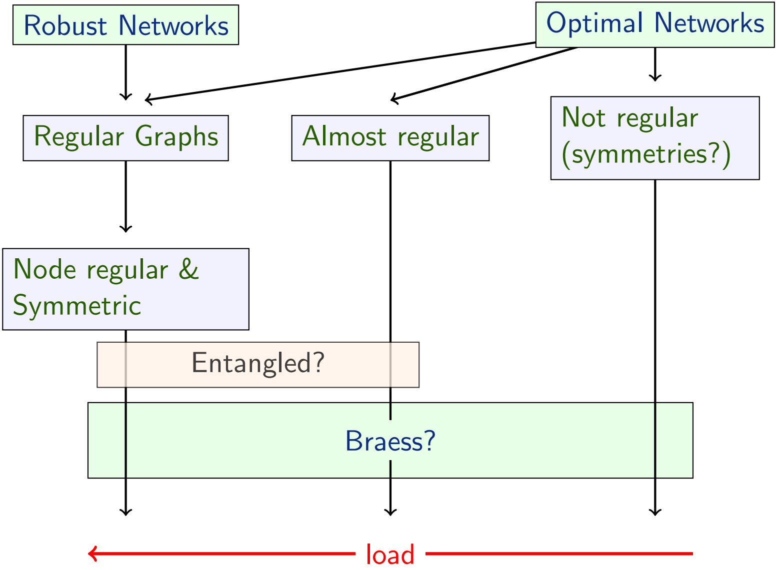
# Outline


Robust Networks
Optimal Networks
Regular Graphs
Almost regular
Not regular (symmetries?)
Node regular & Symmetric
Entangled?
Braess?
load

# Optimal Network

We are interested in how to deliver efficiently information in a communications network (e.g. minimising the transit time of information (packets))

## Possible Approaches

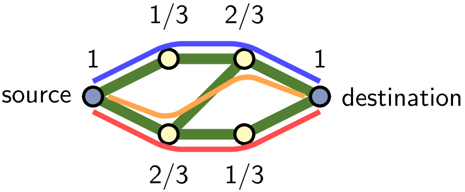
- Given a network 'find' an algorithm to optimise the delivery of packets
- Given a packet delivering algorithm 'build' a network that is optimal for this algorithm

R. Guimerà *et al.*, Optimal network topologies for local search with congestion, Physical Review Letters,**89**, 2002

# The Network

- Fixed number of nodes and links.
- Each node is a source of traffic and has a queue (M/M/1).
- Each node produces the same amount of traffic.
- Estimate the traffic load at node $i$ using the Betweenness Centrality (assumption: routing using shortest–path)

$$\text{Betweenness} = \mathcal{C}_B = \frac{\text{number of shortest–paths that visit a node}}{\text{number of different shortest paths}}$$



# Delay and Congestion

Total number of packets on the network, $n(t)$: (from Little's law)

$$\frac{d\,n(t)}{d\,t} = \Lambda N - \frac{n(t)}{\bar{\tau}}.$$

$N =$ number of nodes, $\Lambda =$ average traffic per node, $\bar{\tau}=$ average delay.

$$\Lambda N \to \text{traffic going in}$$

$$n(t)/\bar{\tau} \to \text{traffic going out}$$

For low load $\Lambda << 1$, $\bar{\tau} \approx$ average shortest–path.

# Delay and Congestion

Average time

$$\bar{\tau} = \frac{1}{N} \sum_{i=1}^{N} \bar{W}_i$$

Average time that a packet spends in queue $i$.

$$\bar{W}_i = \rho_i/(1-\rho_i)(1/\mu), \quad \text{where} \quad \rho_i = \lambda_i/\mu$$

$\mu =$ service rate

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Steady state solution $d\,n(t)/d\,t = 0$ gives:

$$\bar{n} = \sum_{i=1}^{N} \frac{\Lambda(N-1)}{\mu_i(N-1) - \Lambda \mathcal{C}_B(i)}.$$

Congestion (queue node $m$)

$$\boxed{\Lambda_c = (\mu(N-1))/\mathcal{C}_B(m)}$$

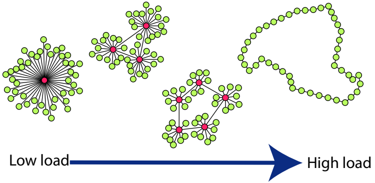
Zhao *et al.*, Physical Review E, **71**, 2005

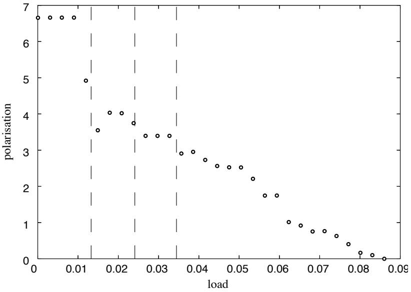
# Re–wiring the Network

- Given a load $\Lambda$
- the number of nodes $N$ and links $L$
- find the network with minimum average delay (minimise $\bar{n}$)

  The rewiring is done using simulated annealing

Polarisation $= (\ell^* - \ell)/\ell$, $\ell^*$ average shortest path for largest congestion load, $\ell$ is average shortest path




number of nodes $=$ number of links

# Some Properties

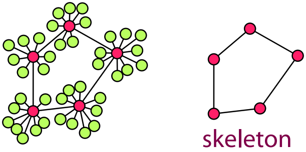
If $N$ number of nodes equal to $L$ the number of links then

- we can evaluate analytically the betweenness, if the graph has $S$ ‘stars’

$$\mathcal{C}_B(ray) = N - 1$$

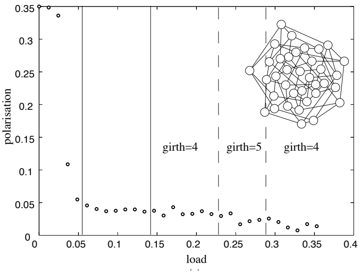
$$\mathcal{C}_B(centre) = \frac{\mathcal{C}_B(SK)N^2 - NS + N^2S + S^2 - NS^2}{S^2}$$

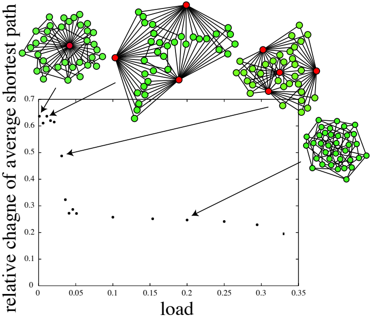
where $\mathcal{C}_B(SK)$ is the betweennes of the skeleton graph.




- we can evaluate the optimal networks as a function of the load
- Transition: 1–star ⇒ 3–star ⇒ 5–star ⇒ 7–star ...

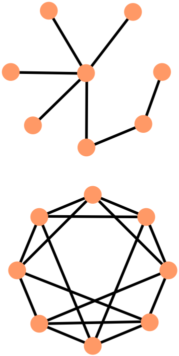
# From Stars to Regular Graphs




2(number of nodes) = number of links

# Robust Networks



Why?

- Increasing use of networks as infrastructure (e–commerce)
- Increase threat of disruption of the communications due to failure or attacks (lack of robustness).

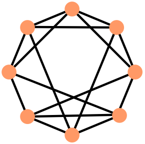
## Solution

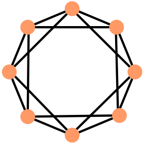
All the nodes 'look' the same so no node is special.

Removing one node will not disrupt the 'flow of information'.

A. H. Dekker & B. D. Colbert, Network Robustness and Graph Topology, 27th Australasian Computer Science Conference, 2004

# Robust Networks





- Node connectivity $= \kappa$: Minimum number of nodes needed to remove to obtain a disconnected network
- Link connectivity $= \eta$: Minimum number of links needed to remove to obtain a disconnected network.
- $d_{min}$: minimum degree in the graph
- Any graph: $\kappa \leq \eta \leq d_{min}$,

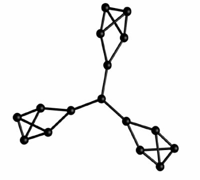
Robust Networks (Dekker & Colbert): They are regular graphs with $\kappa = \eta = d_{min} = d$

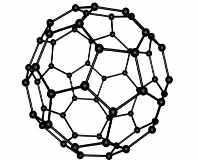
# Regular and Symmetric Graphs

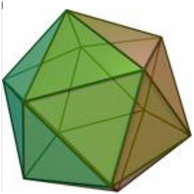
**Theorem**

*(Dekker & Colbert) For any connected node–similar graph of degree d:*

1. $\eta = d$ *(link connectivity = degree)*
2. *if* $d \leq 4$, *then* $\kappa = d$ *(node connectivity = degree)*
3. *if the graph is symmetric, then* $\kappa = d$
4. $\kappa \geq 2/3(d + 1)$



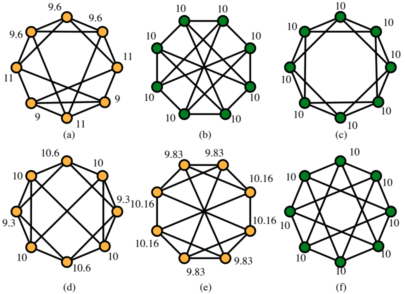




regular → node–similar → symmetric

# Which Regular Graphs?

- These graphs are all regular




- The sum of the betweenness is the same $\sum_i \mathcal{C}_B(i) = 80$
- The average shortest path is the same.
- but the nodes congest at different loads
- the girth is different

# Removing Nodes in an Optimal Network

The load in the links

Symmetric networks:

$$load\ link = \frac{(N-1)\bar{\ell}}{k} \geq \frac{ND}{2k}$$

Node similar networks:

$$load\ link_{max} \geq \frac{(N-1)\bar{\ell}}{k} \geq \frac{ND}{2k}$$

where

- $N =$ number of nodes
- $k =$ degree of the nodes
- $\bar{\ell} =$ average shortest–path
- $D =$ diameter of the network (largest shortest–path)

# Entangled Networks. Synchronisation

Very briefly (from review by Donetti *et al.* )

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = F(x_i) - \sigma \sum_j L_{ij} H(x_j)$$

$F(x_i)$ describes the evolution, $H(x_j)$ the coupling between neighbours and $\sigma$ is a constant.
$L_{ij}$ is the Laplacian matrix

$$L_{ij} = \begin{cases} -1 & \text{if there is a link between i and j} \\ k_i & \text{if j=i, and } k_i \text{ is the degree of node i} \\ 0 & \text{if there is no link between i and j} \end{cases}$$

L. Donetti *et al.* Optimal network topologies: Expanders, Cages, Ramanujan graphs, Entangled networks and all that. arXiv:cond:mat/0605565

# Entangled Networks. Synchronisation

'Robust' synchronised state if the ratio $Q = \lambda_N/\lambda_2$ is as small as possible; $\lambda_i$ = eigenvalue of $L$. (Barahona and Pecora, Wang and Chen)

Properties:

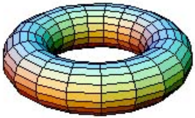
- Homogeneous regular networks (entagled networks Donetti *et al*)
- long loops (large girth)

But synchronisation is not necessary a property wanted in communication networks (route flapping).

Large girth means that if a link fails, there is a long detour when delivering the information

# Adding Links

- In a rectangular–toroidal network the addition of a small amount of random links reduces the value of the critical load in–spite of the increased connectivity between the nodes (Fukś and Lawniczak)

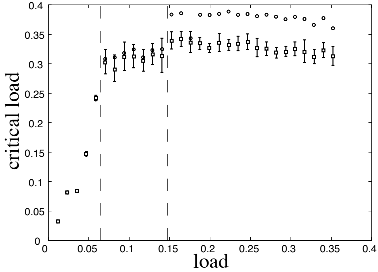


**Braess' paradox:** Each user chooses to minimise its expected delay by choosing an 'optimal' route. The addition of an extra link and hence route choice could reduce the delay. This is true for uncongested networks but it may not be true for a congested network.

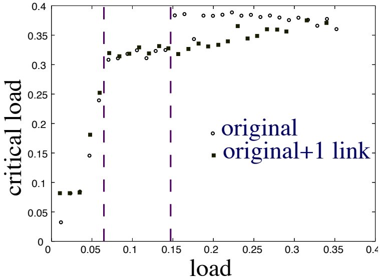
D. Braess, Unternehmensfoschung, 12:258-268, 1968

Networks with Qs, Kelly *et al*, Calvert *et al*

# Adding Links. Braess' Paradox




adding one link




adding one link and then optimising

# More Questions and Some Conclusions

- Low loads: For simple networks ($L = N$) it can be solved, can we use symmetry (group Th.) to obtain (an approximation of) the optimal solution?
- 'middle' of the range loads: Look like entangled networks, are they?
  - desirable qualities: adding new links has a small effect on the performance of the network (Braess).
- High loads: They seem to be node–regular networks (or even Symmetric).
  - Girth changes with the load and is relatively large (perhaps not a good characteristic in communication networks).

# Possible future work

- Nodes that are not equialent (fast queues).