Connectivity of random 1-dimensional networks

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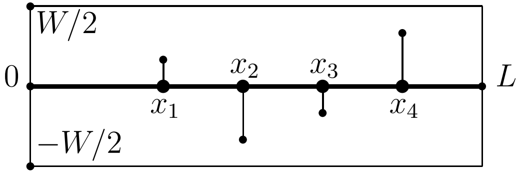
Initial motivations

- dimension 1: monitoring roads, boundaries of restricted areas
- random: automatic deployment along riversides difficult of access

Distributing along roads

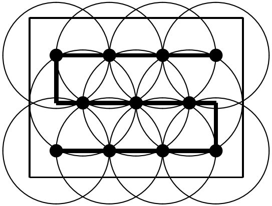
Transmission radius $R >$ road width W.

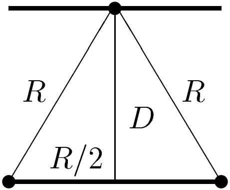
Then a 2-dim network of (x_i, z_i) is connected iff the 1-dim network of x_i with radius $\sqrt{R^2 - W^2}$ is connected.



Filling a 2-dimensional area

Distributing sensors along a snake-like path fills an area if the distance between adjacent branches $D \leq R\sqrt{3}/2$.





What is random?

- common: all (positions of) sensors have a prescribed density function
- practical: deploy sensors one by one along a trajectory of a vehicle, so the distance between successive sensors has a prescribed density

Our assumptions

- R is a transmission radius
- sensors are deployed in $[0, L]$, a sink node is fixed at $x_0 = 0$
- $f_1, \ldots, f_n$ are independent densities of distances between sensors: $P(0 \leq x_i - x_{i-1} \leq R) = \int_0^R f_i(s)ds$.

Connectivity and coverage

For a given probability and densities

- find a minimal number of randomly deployed sensors in $[0, L]$ such that the resulting network is connected;
- find a minimal number of random sensors such that the network is connected and covers $[0, L]$.

Key steps of our solution

- For arbitrary densities $f_1, \ldots, f_n$, compute the probability P_n that the network of n sensors is connected.
- Find estimates of n such that P_n is greater than the given probability.

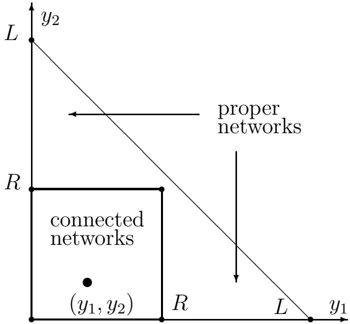
Conditional probabilities

Given densities $f_1, \ldots, f_n$ of distances, $y_1, \ldots, y_n$ are naturally defined on $[0, L]$, but the network should be proper, i.e. all sensors are in $[0, L]$ or $\sum_{i=1}^{n} y_i \leq L$.

We compute the probability that the network is connected if it is proper.

2-sensor networks

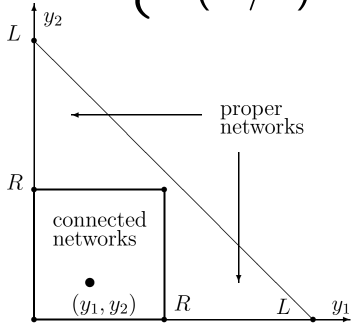
A network of 2 sensors with distances $y_1 = x_1 - 0$, $y_2 = x_2 - x_1$ is represented by $(y_1, y_2) \in \{y_1, y_2 \geq 0\ y_1 + y_2 \leq L\}$.



Simplest non-trivial case

The probability of connectivity is

$$P_2^u = \begin{cases} 2(R/L)^2 & \text{if } R \leq L/2, \\ 4(R/L) - 2(R/L)^2 - 1 & \text{if } R \geq L/2. \end{cases}$$



Connectivity Theorem

The probability of connectivity is

$$P_n = v_n(R, L)/v_n(L, L), \text{ where}$$

$$v_0(r, l) = 1,\ r, l > 0;$$

$$v_n(r, l) = 0,\ r \le 0 \text{ or } l \le 0;$$

$$v_n(r, l) = 1,\ r \ge l > 0,\ n > 0;$$

$$v_n(r, l) = \int_0^r f_n(s) v_{n-1}(r, l - s) ds,\ r < l.$$

$$P_n = v_n(R, L)/v_n(L, L)$$

- $+$ closed formula for finite networks
- $+$ arbitrary different densities
- $-$ can be computationally difficult
- $+$ explicit for important distributions
- $+$ implies simple estimates for n

The recursive function

$v_n(r, l)$ is the probability that random distances having densities $f_1, \ldots, f_n$ satisfy $\sum_{i=1}^{n} y_i \leq l$ and $0 \leq y_i \leq r$, e.g.

$v_1(r, l) = \int_0^r f_1(s)ds$, $r < l$,

$v_2(r, l) = \int_0^r f_2(s)v_1(r, l - s)ds$.

$v_n(L, L)$: the network is proper,

$v_n(R, L)$: the network is connected.

Coverage Theorem

The probability of coverage is

$$(v_n(R, L) - v_n(R, L - R))/v_n(L, L).$$

$\frac{v_n(R,L)}{v_n(L,L)}$: connected if proper on $[0, L]$,

$v_n(R, L - R)/v_n(L, L)$: connected network if proper on $[0, L - R]$.

Uniform Corollary

If all $f_i = 1/L$ then the probability is

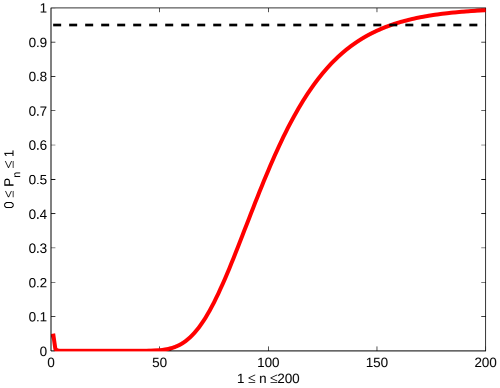
$$P_n^u = \sum_{i=0}^{i<L/R} (-1)^i \binom{n}{i} (1 - iR/L)^n.$$

$P_1^u = R/L$: connected with the sink.

$$P_2^u = \begin{cases} 2(R/L)^2 & \text{if } R \leq L/2, \\ 4(R/L) - 2(R/L)^2 - 1 & \text{if } R \geq L/2. \end{cases}$$

Uniform case: simulations

$L = 1\text{km}$, $R = 50\text{m}$, $n \leq 200$ sensors.



Uniform case: estimate

Set $Q = (L/R) - 1$. The network is connected with a probability $p > 2/3$ if

$$n \geq \frac{3}{2}(1-Q)+\sqrt{\frac{(3Q-1)^2}{4}+6Q^2\left(\frac{Q}{1-p}-1\right)}.$$

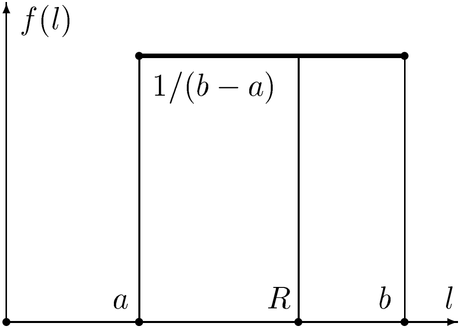
Transmission Radius, m.	200	100	50	25
Min Number of Sensors	29	69	157	349
Estimate of Min Number	83	283	905	2610

Uniform case: conclusions

- $-$ less effective than non-random
- $-$ rough estimate, not optimal
- $+$ quadratic estimate is used later
- $+$ can be improved using Taylor approximations of degrees 4, 5
- $+$ non-trivial inequalities $0 \leq P_n^u \leq 1$

A constant density: graph

Let $f = 1/(b-a)$ over $[a, b] \subset [0, L]$.



$n = 1$: $P(0 \leq y_1 \leq R) = (R-a)/(b-a)$.

Constant Corollary

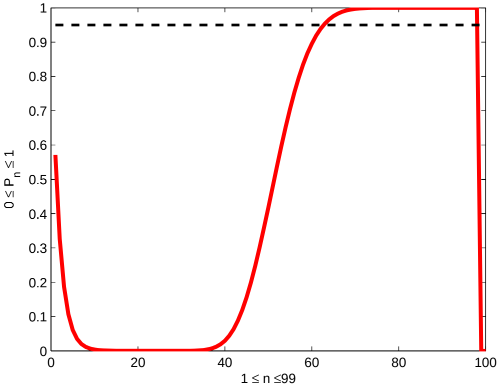
If all $f_i = 1/(b-a)$ then the probability is

$$P_n^c = \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} (L - a(n-k) - Rk)^n}{\sum_{k=0}^{n} (-1)^k \binom{n}{k} (L - a(n-k) - bk)^n}.$$

$$P_1^c = \frac{(L-a)-(L-R)}{(L-a)-(L-b)} = \frac{R-a}{b-a}.$$

Constant case: simulations

$L = 1\text{km}, R = 50\text{m}, a = 10\text{m}, b = 80\text{m}.$



Constant case: estimate

The network is connected with a probability p if $\frac{a+b}{2} \leq R \leq b$ and

$$n \geq \max\left\{\frac{3}{2} + \sqrt{\frac{1+5p}{1-p}},\ 1 + \frac{L-b}{a}\right\}.$$

For all p not too close to 1, the 2nd estimate holds: $L + a - b \leq an < L$.

Constant case: conclusions

Constant density over $[0.2R, 1.6R]$.

Transmission Radius, m.	200	150	100	50	25
Min Number of Sensors	14	19	30	63	132
Estimate of Min Number	18	27	43	93	193
Max Number of Sensors	25	34	50	100	200

- $+$ minimal practical assumptions
- $+$ very simple effective estimate
- $+$ non-trivial inequalities $0 \leq P_n^c \leq 1$

Exponential Corollary

If the distances between successive sensors have the density $f(s) = ce^{-\lambda s}$ on $[0, L]$, then the probability of connectivity is $P_n^e = \frac{v_n(R, L)}{v_n(L, L)}$, $v_n(r, l) =$

$$\sum_{i=0}^{i<l/r} (-1)^i \binom{n}{i} \frac{e^{-i\lambda r}}{\lambda^n} \left(1 - e^{-\lambda(l-ir)} \sum_{j=0}^{n-1} \frac{\lambda^j (l-ir)^j}{j!} \right).$$

Exponential conclusions

Estimate: as in the uniform case.

The denominator tends to 0 fast:

$$v_n(L, L) = 1 - e^{-\lambda L} \sum_{j=0}^{n-1} (\lambda L)^j / j!$$

- unpractical: throw on the alert
- sensors are too close to each other

Normal distribution

If $f(s) = \frac{c}{\sigma\sqrt{2\pi}} e^{-(s-\mu)^2/2\sigma^2}$ on $[0, L]$

then the distances between successive sensors are close to the mean μ, e.g. very likely to be in $[\mu - 3\sigma, \mu + 3\sigma]$

Reasonable to assume: $\mu < R$, $n\mu < L$.

Normal case: estimate

The network with normal distances is connected with a given probability p if

$$n \le \min\left\{ \frac{p(1-p)}{\varepsilon}, \frac{(\sqrt{4\mu L + \sigma^2\Phi^{-2}(p)} - \sigma\Phi^{-1}(p))^2}{4\mu^2} \right\}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-s^2/2} ds, \ \varepsilon = \Phi\left(-\frac{\mu}{\sigma}\right) + 1 - \Phi\left(\frac{R-\mu}{\sigma}\right).$$

Normal case: example

$\mu = 0.6R$, $\sigma = 0.1R$, $p = 0.9975$.

Then $\Phi^{-1}(p) \approx 2.8$, $\varepsilon \approx 0.000063$.

$R = 25$m: $n \leq p(1-p)/\varepsilon \approx 40$.

$R \geq 50$m: the 2nd estimate is close to

$$L/\mu \approx \frac{(\sqrt{4\mu L + \sigma^2\Phi^{-2}(p)} - \sigma\Phi^{-1}(p))^2}{4\mu^2}.$$

Normal case: table

Let $L = 1$km, $\mu = 0.6R$,$\sigma = 0.1R$.

Transmission Radius, m.	200	150	100	50	25
Estimate of Max Number	7	11	16	33	40

6 non-random sensors are enough for the radius $R = 150$m: $6/11 \approx \mu/R$.

All cases: conclusions

- – exponential: too dense networks
- – normal: ideal density $\Rightarrow$ ideal results
- – uniform: a useful theoretical exercise
- + constant over $[a, b]$: very reasonable
- + more complicated: piecewise constant?

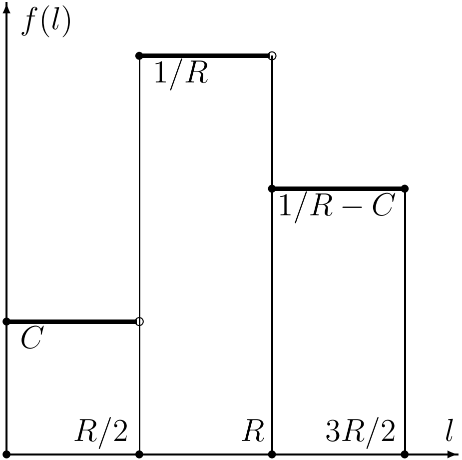
Ideas of proofs

- induction on the number of sensors: adding 1 sensor keeps connectivity if it is close to the previous one
- the key probability $v_n(r, l)$ is an iterated convolution of densities computed by the Laplace transform

More explicit formulae

- heterogeneous networks: distances have different constant densities
- building densities from blocks: any piecewise constant density
- more can be produced easily

A 3-step density: graph



C, R are chosen so that $\int_0^L f(s)ds = 1$.

A 3-step density: formula

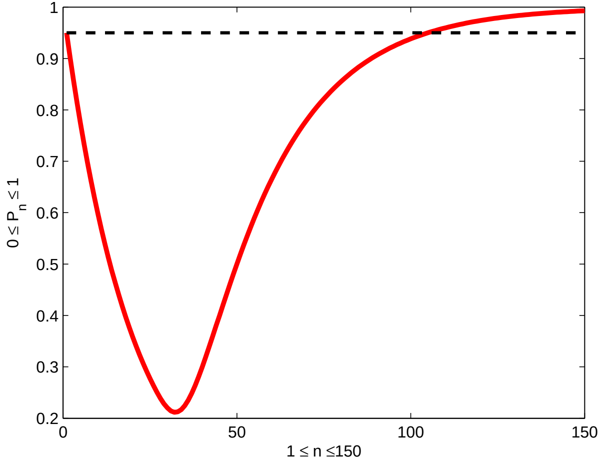
The probability of connectivity is $P_n =$

$$\frac{\displaystyle\sum_{m=0}^{n}\sum_{k_1=0}^{m}\sum_{k_2=0}^{n-m}\frac{(-1)^{k_1+k_2}(L-(2k_1+k_2+n-m)R/2)^n}{d_m k_1!(m-k_1)!k_2!(n-m-k_2)!}}{\displaystyle\sum_{m=0}^{n}\sum_{k_1=0}^{m}\sum_{k_2=0}^{n-m}\frac{(-1)^{k_1+k_2}(L-(2k_1+2k_2+n-m)R/2)^n}{d_m k_1!(m-k_1)!k_2!(n-m-k_2)!}}$$

$d_m = C^{-m}(1/R - C)^{m-n}$, the sums are over all m, k_1, k_2 if the terms > 0.

A 3-step density: simulations

Let $L = 1\text{km}$, $R = 50\text{m}$, $C = 0.9/R$.



A 3-step density: table

Transmission Radius, m.	250	200	150	100	50
Min Number of Sensors	12	17	25	44	105

+ flexible practical assumptions
+ reasonable estimates of min number
+ non-trivial inequalities $0 \leq P_n \leq 1$

Open problem 1

Compute the exact probability of connectivity if the distances between successive sensors have a truncated normal density on $[0, L]$.

Open problem 2

For a given segment $[0, L]$ and number n of sensors, find an optimal density of distances between successive sensors to maximise the probabilities of connectivity and coverage.

Open problem 3

Compute the probabilities of connectivity and coverage if sensors are randomly deployed along a non-straight trajectory filling a 2-dimensional area.